INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

On the Approximation of the Area of a Surface

Jean-Marie Morvan — Boris Thibert

N° 4375

Février 2002

THÈME 2

Rapport de recherche

ISSN 0249-6399

On the Approximation of the Area of a Surface

Jean-Marie Morvan , Boris Thibert

Thème 2 — Génie logiciel
et calcul symbolique
Projet Prisme

Rapport de recherche n° 4375 — Février 2002 — 9 pages

Abstract: We compare the area of a smooth surface S with the area of a surface M which is *close to it*, in terms of the local structure of S and the distance between S and M. We obtain convergence results, in particular when the surface approaching S is the restricted Delaunay complex associated to an ϵ-sample $\mathcal{S}$ of S.

Key-words: Surface, area, curvature, Hausdorff distance, Delaunay triangulation, sample.

Work partially supported by the IST Programme of the EU as a Shared-cost RTD (FET Open) Project under Contract No IST-2000-26473 (ECG - Effective Computational Geometry for Curves and Surfaces) and by Région Rhônes-Alpes (France), Contrat 99029744.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Sur l'approximation de l'aire d'une surface

Résumé : Nous comparons l'aire d'une surface lisse S et l'aire d'une surface M qui lui est proche, au moyen de la structure locale de S et de la distance entre S et M. Nous obtenons des résultats de convergence, en particulier lorsque la surface qui approche S est le complexe de Delaunay restreint associé à un ϵ-échantillonage $\mathcal{S}$ de S.

Mots-clés : Surface, aire, courbure, distance de Hausdorff, triangulation de Delaunay, échantillonage.

1 Introduction

Consider a smooth (compact orientable) surface S of the Euclidean space $\mathbb{E}^3$, and a subset M, *"close to S"* smooth almost everywhere. When does the area $\mathcal{A}(M)$ of M approximate the area $\mathcal{A}(S)$ of S?

Before dealing with this problem, we must remark the following:

- Instead of surfaces, consider a curve C in $\mathbb{E}^3$ and a sequence of polygons P_n whose vertices lie in C, and whose initial and final points are the initial and final points of C. If the length of the edges of the polygons tends to zero, then *by definition of the length*, the length of C is the limit of the length of P_n.

- The situation is completely different for surfaces: the well known *"Lampion de Schwarz"* shows that the area of a sequence of triangulations inscribed in a fixed cylinder can tend to infinity. In order to describe succintly this phenomena, let us build a 2-parameter family of *generalized Lampions de Schwarz.* Let C be a cylinder of finite height H and radius R in $\mathbb{E}^3$. Let $P(n, N)$ denote the triangulation inscribed in C defined as follows: consider N circles on the cylinder C obtained by intersecting C by 2-planes orthogonal to the axis of C. Inscribe on each circle a regular n-gon such that the n-gon on the slice k is deduced from the n-gon of the slice $k-1$ by a rotation of angle $\frac{2\pi}{n}$. Then join each vertex v of the slice $k-1$ to the two vertices of the slice k which are on each side of v. We obtain a triangulation whose vertices $v_{i,j}$ are defined as follows:

$$\begin{array}{ll} \forall i \in \{0, ..n-1\}, & v_{i,j} = (R\ \cos(i\alpha), R\ \sin(i\alpha), jh) \text{ if } j \text{ is even,} \\ \forall j \in \{0, ..N\}, & v_{i,j} = (R\ \cos(i\alpha + \frac{\alpha}{2}), R\ \sin(i\alpha + \frac{\alpha}{2}), jh) \text{ if } j \text{ is odd,} \end{array}$$

and whose faces are:

$$\begin{array}{l} v_{i,j}\ v_{i+1,j}\ v_{i,j+1}, \\ v_{i,j}\ v_{i-1,j+1}\ v_{i,j+1}, \end{array}$$

where $\alpha = \frac{\pi}{n}$ and $h = \frac{H}{N}$.

Then when n tends to infinity, the area $\mathcal{A}(P(n, n^3))$ of $P(n, n^3)$ tends to infinity.

J. Fu [6] obtained a result of convergence of the area of a sequence of triangulations which converges to a smooth surface. In this short note, we obtain a general result: we give an upper bound and a lower bound of the area of S in terms of the area of M and geometric quantities related to S, as soon as M is *"close enough"* to S, in a sense which will be precised in a moment. (In the case where M is a triangulation, we do not need every angle of M to be quite large -compare with [9]-) . We give an application of this result in the theory of surface reconstruction from scattered sample points: if $\mathcal{S}$ is a ϵ-sample of a (closed) surface, (in the sense of [1], [3]), then the limit of the area $\mathcal{A}(T_\epsilon)$ of the Delaunay triangulation T_ϵ satisfies:

$$\lim_{\epsilon \to 0} \mathcal{A}(T_\epsilon) = \mathcal{A}(S).$$

2 The General Framework

2.1 Smooth surfaces

In the following, by a *smooth surface* S we mean a compact (oriented) regular $\mathcal{C}^2$ surface with or without boundary ∂S, embedded in $\mathbb{E}^3$. S is endowed with the Riemannian structure induced by the standard scalar product of $\mathbb{E}^3$. We denote by da_S the area form on S and by ds the canonical orientation of ∂S. Let ν be the unitary normal vector field on S (compatible with the orientation of S). Since S is $\mathcal{C}^2$, we can define the second fundamental form h of S in the direction ν, and the Weingarten endomorphism A_p (which is the adjoint of h_p). At each point p of S, we denote by λ_p^1, λ_p^2 the principal curvatures of S at p, that is, the eigenvalues of h_p. The determinant of h_p is *the Gauss curvature* G_p of S at p, its trace is *the mean curvature* H_p of S at p. *The maximal curvature of* S *at* p is $\rho_p = max(|\lambda_p^1|, |\lambda_p^2|)$. The function ρ is continuous function on S. For more details on smooth surfaces, one may refer to [4] or [2].

To compare the area of two surfaces close from one another (in a sense that we shall precise later), we need to introduce the projection function ξ: let m be any point of $\mathbb{E}^3$, and consider the subset of points of S whose distance to m is minimum. If S is compact, this subset is not empty. If it is reduced to a single point, we call it $\xi(m)$, and then define the projection function ξ.

The existence of ξ and its regularity have been extensively studied. The following result (see [5]) ensures that, when S is smooth, ξ is well defined in a small neighborhood of S.

Proposition 1 *Let S be a smooth compact surface of $\mathbb{E}^3$. Then there exists an open set U_S of $\mathbb{E}^3$ containing S where ξ is well defined. Moreover ξ is continuous in U_S.*

The open set U_S depends both locally and globally on the smooth surface S. Globally, U_S depends on points which can be far from one another on the surface, but which are close in $\mathbb{E}^3$. Locally, the normals of S do not intersect in U_S.

For the following, we need the

Definition 1 *Let S be a smooth surface of $\mathbb{E}^3$.*

- *The function $\omega_S : U_S \to \mathbb{E}$ defined by*

$$\omega_S(m) = ||\xi(m) - m||\rho_{\xi(m)}$$

  is called the relative curvature function of S on U_S.

- *The real number*

$$\omega_S(M) = \sup_{m \in M} \omega_S(m),$$

  is called the relative curvature of M with respect to S.

We shall also need the notion of *reach of a surface*, introduced in [5].

Definition 2 *The reach r of a surface S is the largest $r > 0$ for which ξ is defined on the (open) tubular neighborhood $U_r(S)$ of radius r of S.*

Remark in particular that if M lies in $U_r(S)$, then $\omega_S(M) \leq 1$, (see [8] for instance). (The notion of reach can be related to the notion of *local feature size* -LFS-, (see [1] and [3]), but we don't need this relation in our context. Finally, we give the following

Definition 3 *A subset M of $\mathbb{E}^3$ is closely near S if it lies in $U_r(S)$, where r is the reach of S and if the restriction of ξ to M is one-to-one.*

If M is closely near S and differentiable almost everywhere, one can define at almost every point m of M the angle $\alpha_m \in \left[0, \frac{\pi}{2}\right]$ between the tangent planes $T_m M$ and $T_{\xi(m)}S$. We put

$$\alpha_{\max} = \sup_{m \in M} \alpha_m,$$

$$\alpha_{\min} = \inf_{m \in M} \alpha_m.$$

3 The Approximation of the Area

The following result shows that if M is closely near S and has enough regularity to have a tangent plane almost everywhere, then the area of S is bounded from above and from below by quantities depending on **the area of M, the Hausdorff distance between M and S, the curvature of S, and the angle between the corresponding tangent planes**.

Theorem 1 *Let S be a (compact orientable) C^2 surface in $\mathbb{E}^3$. Let M be a surface which is differentiable almost everywhere, and closely near S. Then the area satisfies:*

$$\mathcal{A}(S) = \int_M \frac{\cos \alpha_m}{1 + \delta_m \epsilon_m H_{\xi(m)} + \delta_m^2 G_{\xi(m)}} da_M(m),$$

where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} >\in \{-1, +1\}$ and $\delta_m = \|\xi(m) - m\|$.

As an obvious consequence, one has the following

Corollary 1 *Let M be a surface closely near S. Then,*

$$\frac{\cos \alpha_{\max}}{(1 + \omega_S(M))^2} \mathcal{A}(M) \leq \mathcal{A}(S) \leq \frac{\cos \alpha_{\min}}{(1 - \omega_S(M))^2} \mathcal{A}(M).$$

The Hausdorff distance δ_{Hauss} between two subsets A and B of $\mathbb{E}^3$ is defined by

$$\delta_{Hauss}(A, B) = max(\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)).$$

We deduce from the previous corollary the following

Corollary 2 *Let S be a (compact orientable) C^2 surface in $\mathbb{E}^3$. Let M_n be a sequence of surfaces closely near S which are differentiable almost everywhere. If*

1. *the Hausdorff limit of M_n is S, when n goes to infinity,*
2. *the angle between tangent planes $T_m M_n$ and $T_{\xi(m)}S$ tends to 0 almost everywhere, when n goes to infinity,*

then,

$$\lim_{n\to\infty} \mathcal{A}(M_n) = \mathcal{A}(S).$$

We present now two examples showing the relevance of the introduced geometric quantities.

1. First of all, consider two spheres S and M with same centers and different radius 1 and R. The angle between the normals of S and M is 0, but the relative curvature of M with respect to S is equal to $1-R$. If R tends to 0 then $\omega_S(M)$ tends to 1 and the area of M to 0.

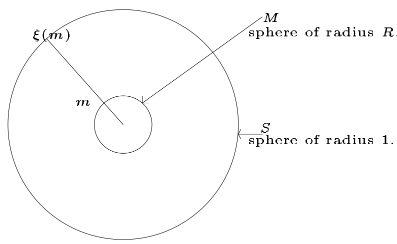

Figure 1: $\cos\alpha_{max} = 1$ and $\omega_S(M)$ is large

2. Consider now a flat surface S, and a sequence $(M_n)_{n\geq 1}$ of triangulations such that $\omega_S(M_n)$ is equal to 0 and α_{max} is equal to $\frac{\pi}{4}$. The area of M_n is equal to $\sqrt{2}$ times the area of S.

4 An application in Surface Reconstruction

Let $\mathcal{S}$ be a finite set of points inscribed in a (compact) surface S. We remind the following

Definition 4 *The set T of Delaunay simplices of $\mathcal{S}$ whose dual intersects S is called the* **restricted Delaunay complex** *of $\mathcal{S}$ with respect to S.*

If the sample has suitable properties, then T has interesting topological and geometrical properties. We will use the concept of ϵ-sample, introduced in [1]:

Definition 5 *A set of points $\mathcal{S}$ on S is an ϵ-sample of S if and only if for every point m of S, the ball $B(m, \epsilon LFS(m))$ encloses at least one point of $\mathcal{S}$.*

Let us summarize the main properties of the restricted Delaunay complex associated to an ϵ-sample, [1].

Proposition 2 *Let S be a compact surface of $\mathbb{E}^3$. Let $T_{\mathcal{S}_\epsilon}$ be the restricted Delaunay complex of an ϵ-sample $\mathcal{S}_\epsilon$ with respect to S.*

1. *If $\epsilon < 0.1$, then $T_{\mathcal{S}_\epsilon}$ is homeomorphic to S.*
2. *The maximum angle $\alpha_{\max}$ between the corresponding tangent planes of $T_{\mathcal{S}_\epsilon}$ and S (via ξ) satisfies:*
$$\alpha_{\max} = O(\epsilon).$$

If ϵ is small enough, it is easy to check that the map ξ is well defined and one-to-one. As an immediate consequence of Proposition (2) and Theorem (1), we get the following

Corollary 3 *Let S be a compact surface of $\mathbb{E}^3$. Let $T_{\mathcal{S}_\epsilon}$ be the restricted Delaunay complex of an ϵ-sample $\mathcal{S}_\epsilon$ with respect to S. Then*
$$\lim_{\epsilon \to 0} \mathcal{A}(T_{\mathcal{S}_\epsilon}) = \mathcal{A}(S).$$

5 Proof of Theorem 1

The following proposition gives the main differential properties of the function ξ when it is defined. It has been studied in [7] in the case where S is the boundary of a convex set of $\mathbb{E}^3$ and in [9].

Proposition 3 *Let S be a smooth surface of $\mathbb{E}^3$ without boundary, U_S an open subset of $\mathbb{E}^3$ where the map $\xi : U_S \to S$ is well defined. Then,*

1. *the map ξ is C^1 in U_S and satisfies for every $m \in U_S$:*
$$D\xi(m)(Z_m) = 0, \forall Z_m \text{ orthogonal to } T_{\xi(m)}S,$$
$$D\xi(m)(X_m) = (Id + \delta_m \epsilon_m A_{\xi(m)})^{-1}(X_m), \forall X_m \text{ parallel to } T_{\xi(m)}S,$$
where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} >\in \{-1,+1\}$ and $A_{\xi(m)} = -D\nu(\xi(m))$ is the Weingarten endomorphism of S at the point $\xi(m)$.
2. *In particular, the matrix of $D\xi(m) : \mathbb{E}^3 \to T_{\xi(m)}S$ (in local orthonormal frames $(e^1_{\xi(m)}, e^2_{\xi(m)}, \nu^S_{\xi(m)})$ and $(e^1_{\xi(m)}, e^2_{\xi(m)})$) is given by:*
$$\begin{pmatrix} \frac{1}{1+\delta_m \epsilon_m \lambda^1_{\xi(m)}} & 0 & 0 \\ 0 & \frac{1}{1+\delta_m \epsilon_m \lambda^2_{\xi(m)}} & 0 \end{pmatrix}.$$

where $e^1_{\xi(m)}$ and $e^2_{\xi(m)}$ are unitary principal vectors and $\nu^S_{\xi(m)}$ is the oriented normal of S at $\xi(m)$.

Sketch of proof of Proposition 3

- For every $m \in U_S$, the point $\xi(m)$ is defined by the following relation:

$$\forall m \in U_S, \forall X \in T_{\xi(m)}S, < \xi(m) - m, X_{\xi(m)} >= 0.$$

  Consequently, for every $m \in T_m S$, the function ξ is constant on the orthogonal of $T_m S$.

- Consider now a vector $X_m \in T_m U_S$ which is parallel to $T_{\xi(m)}S$. We have:

$$D\xi(m)(X) = X + \delta_m \epsilon_m D\nu(\xi(m)) \circ D\xi(m)(X).$$

  The endomorphism $(I + \delta_m \epsilon_m A_{\xi(m)}$ is clearly invertible, and consequenly,

$$D\xi(m)(X) = (I + \delta_m \epsilon_m A_{\xi(m)})^{-1}(X).$$

- The rest of the proof is obvious.

Now consider the surface M lying in U_S. By assumption, the restriction $\xi_{|M}$ of ξ to M is one-to-one, and then is a diffeomorphism. We shall apply the classical area formula:

$$\mathcal{A}(S) = \int_S da_S = \int_M \xi^*_{|M} da_S = \int_M |D\xi_{|M}(m))| \, da_S(\xi(m)),$$

which is pertinent since we are in the case where ξ is a diffeomorphism. The following lemma evaluates the Jacobian of $\xi_{|M}$.

Lemma 1 *Let S be a smooth surface of $\mathbb{E}^3$ without boundary, U_S an open subset of $\mathbb{E}^3$ where the map $\xi : U_S \to S$ is well defined. Let $M \subset U_S$ be a smooth surface. Then the Jacobian of the differential of $\xi_{|M}$ is given by:*

$$|D\xi_{|M}(m))| = \frac{cos\alpha_m}{(1 + \delta_m \epsilon_m \lambda^1_{\xi(m)})(1 + \delta_m \epsilon_m \lambda^2_{\xi(m)})},$$

where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} >\in \{-1, +1\}$, and $\delta_m = \|\xi(m) - m\|$.

This lemma is obvious, by considering the restriction $\xi_{|M}$ of ξ to M. Remark that the condition of the theorem on the angle between corresponding tangent planes implies that $\xi_{|M}$ is a diffeomorphism between M and S.

End of the proof of Theorem 1 Since M is closely inscribed in S, $\xi_{|M}$ is one-to-one between M and S. Therefore:

$$\mathcal{A}(S) = \int_M |D\xi_{|M}(m))| \, da_M(m) = \int_M \frac{cos\ \alpha_m}{(1 + \delta_m \epsilon_m \lambda^1_{\xi(m)})(1 + \delta_m \epsilon_m \lambda^2_{\xi(m)})} \, da_M(m).$$

6 Conclusion

In this paper, we obtain an approximation of the area of a surface when this surface is "closed" to another surface whose area is known, using geometric quantities. This is a first step in the study of the approximation of geometric invariants on a surface (the curvatures for instance) with the corresponding invariants of a surface closed to it. In particular, if we deal with smooth surfaces approximated by discrete ones, it would be interesting to have a unified geometric theory including both discrete and smooth objects.

References

[1] N. Amenta, M. Bern, Surface Reconstruction by Voronoi Filtering, Discrete Comput Geom 22:481-504 (1999).

[2] M. Berger, B. Gostiaux, Géométrie différentielle : variétés, courbes et surfaces (Seconde édition, Presses Universitaires de France, Paris, 1992).

[3] D. Attali, J.D. Boissonnat, In preparation.

[4] M. P. Do Carmo, Differential geometry of curves and surfaces (Translated from the Portuguese, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976).

[5] H. Federer, Curvature measures, Trans. Amer. Math. Soc 93 (1959) 418-491.

[6] J. Fu, Convergence of curvatures in secant approximations, J. Differential Geometry 37 (1993) 177-190.

[7] A. Haraux, How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities, J. Math. Soc. Japan 29 N°4 (1977) 615–631.

[8] J. Milnor, Morse theory (Princeton University Press, Princeton, New Jersey, 1963).

[9] J.M. Morvan, B. Thibert, On the approximation of a smooth surface with a triangulated mesh, submitted (2001).

Jean-Marie Morvan:
Institut Girard Desargues and I.N.R.I.A. Sophia Antipolis,

Boris Thibert:
Institut Girard Desargues
Université Claude Bernard Lyon 1,
43 Bd du 11 novembre 1918,
69622 Villeurbanne, Cedex, France.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399